

# BALL THROWING ON SPHERES

ANNE ESTRADÉ AND JACQUES ISTAS

ABSTRACT. Ball throwing on Euclidean spaces has been considered for a while. A suitable renormalization leads to a fractional Brownian motion as limit object. In this paper we investigate ball throwing on spheres. A different behavior is exhibited: we still get a Gaussian limit but which is no longer a fractional Brownian motion. However the limit is locally self-similar when the self-similarity index  $H$  is less than  $1/2$ .

KEYWORDS. fractional Brownian motion, overlapping balls, scaling, self-similarity, spheres

## 1. INTRODUCTION

Random balls models have been studied for a long time and are known as germ-grain models, shot-noise, or micropulses. The common point of those models consists in throwing balls that eventually overlap at random in an  $n$ -dimensional space. Many random phenomena can be modeled through this procedure and many application fields are concerned: Internet traffic in dimension one, communication network or imaging in dimension two, biology or materials sciences in dimension three. A pioneer work is due to Wicksell [24] with the study of corpuscles. The literature on germ-grain models deals with two main axes. Either the research focuses on the geometrical or morphological aspect of the union of random balls (see [20] or [21] and references therein), or it is interested in the number of balls covering each point. This second point of view is currently known as shot-noise or spot-noise (see [6] for existence). In dimension three, the shot-noise process is a natural candidate for modeling porous or granular media, and more generally heterogeneous media with irregularities at any scale. The idea is to build a microscopic model which yields a macroscopic field with self-similar properties. The same idea is expected in dimension one for Internet traffic for instance [25]. A usual way for catching self-similarity is to deal with scaling limits. Roughly speaking, the balls are dilated with a scaling parameter  $\lambda$  and one lets  $\lambda$  go either to 0 or to infinity. We quote for instance [4] and [12] for the case  $\lambda \rightarrow 0^+$ , [11] and [2] for the case  $\lambda \rightarrow +\infty$ , [5] and [3] where both cases are considered.

In the present paper, we follow a procedure which is similar to [2] and [3]. Let us describe it precisely. A collection of random balls in  $\mathbb{R}^n$  whose centers and radii are chosen according to a random Poisson measure on  $\mathbb{R}^n \times \mathbb{R}^+$  is considered. The Poisson intensity is prescribed as follows

$$\nu(dx, dr) = r^{-n-1+2H} dx dr ,$$

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for some real parameter  $H$ . Since the Lebesgue measure  $dx$  is invariant with respect to isometry, so is the random balls model, and so will be any (eventual) limit. As the distribution of the radii follows a homogeneous distribution, a self-similar scaling limit may be expected. Indeed, with additional technical conditions, the scaling limits of such random balls models are isometry invariant self-similar Gaussian fields. The self-similarity index depends on the parameter  $H$ . When  $0 < H < 1/2$ , the Gaussian field is nothing but the well-known fractional Brownian motion [16, 18, 19].

Manifold indexed fields that share properties with Euclidean self-similar fields have been and still are extensively studied (e.g. [8, 9, 13, 14, 15, 17, 22, 23]). In this paper, we wonder what happens when balls are thrown onto a sphere, and no longer onto a Euclidean space. More precisely, is there a scaling limit of random balls models, and, when it exists, is this scaling limit a fractional Brownian field indexed by the surface for  $0 < H < 1/2$ ?

The random field is still obtained by throwing overlapping balls in a Poissonian way. The Poisson intensity is chosen as follows

$$\nu(dx, dr) = f(r) \sigma(dx) dr.$$

The Lebesgue measure  $dx$  has been replaced by the surface measure  $\sigma(dx)$ . The function  $f$ , that manages the distribution of the radii, is still equivalent to  $r^{-n-1+2H}$ , at least for small  $r$ , where  $n$  stands for the surface dimension. It turns out that the results are completely different in the two cases (Euclidean, spherical). In the spherical case, there is a Gaussian scaling limit for any  $H$ . But it is no longer a fractional Brownian field, as defined by [13]. We then investigate the local behavior, in the tangent bundle, of this scaling limit in the spirit of local self-similarity [1, 7, 15]. It is locally asymptotically self-similar, with a Euclidean fractional Brownian field as tangent field. Our microscopic model has led to a local self-similar macroscopic model.

The paper is organized as follows. In Section 2, the spherical model is introduced and we prove the existence of a scaling limit. In Section 3, we study the locally self-similar property of the asymptotic field. Section 4 is devoted to a comparative analysis between Euclidean and spherical cases. Eventually, some technical computations are presented in the Appendix.

## 2. SCALING LIMIT

We work on  $\mathbb{S}_n$  the  $n$ -dimensional unit sphere,  $n \geq 1$ :

$$\mathbb{S}_n = \{(x_i)_{1 \leq i \leq n+1} \in \mathbb{R}^{n+1}; \sum_{1 \leq i \leq n+1} x_i^2 = 1\}.$$

**2.1. Spherical caps.** For  $x, y \in \mathbb{S}_n$ , let  $d(x, y)$  denote the distance between  $x$  and  $y$  on  $\mathbb{S}_n$ , i.e. the non-oriented angle between  $Ox$  and  $Oy$  where  $O$  denotes the origin of  $\mathbb{R}^{n+1}$ . For  $r \geq 0$ ,  $B(x, r)$  denotes the closed ball on  $S$  centered at  $x$  with radius  $r$ :

$$B(x, r) = \{y \in \mathbb{S}_n; d(x, y) \leq r\}.$$

Let us notice that for  $r < \pi$ ,  $B(x, r)$  is a spherical cap on the unit sphere  $\mathbb{S}_n$ , centered at  $x$  with opening angle  $r$  and that for  $r \geq \pi$ ,  $B(x, r) = \mathbb{S}_n$ .

Denoting  $\sigma(dx)$  the surface measure on  $\mathbb{S}_n$ , we prescribe  $\phi(r)$  as the surface of any ball on  $\mathbb{S}_n$  with radius  $r$ ,

$$\phi(r) := \sigma(B(x, r)) , \quad x \in \mathbb{S}_n, r \geq 0 .$$

We also introduce the following function defined for  $z$  and  $z'$ , two points in  $\mathbb{S}_n$  and  $r \in \mathbb{R}^+$ ,

$$(1) \quad \Psi(z, z', r) := \int_{\mathbb{S}_n} \mathbf{1}_{d(x, z) < r} \mathbf{1}_{d(x, z') < r} \sigma(dx) .$$

Actually  $\Psi(z, z', r)$  denotes the surface measure of the set of all points in  $\mathbb{S}_n$  that belong to both balls  $B(z, r)$  and  $B(z', r)$ . Clearly  $\Psi(z, z', r)$  only depends on the distance  $d(z, z')$  between  $z$  and  $z'$ . We write

$$(2) \quad \psi(d(z, z'), r) = \Psi(z, z', r)$$

and note that it satisfies the following:  $\forall (u, r) \in [0, \pi] \times \mathbb{R}^+$

- $0 \leq \psi(u, r) \leq \sigma(\mathbb{S}_n) \wedge \phi(r)$
- if  $r < u/2$  then  $\psi(u, r) = 0$  and if  $r > \pi$  then  $\psi(u, r) = \sigma(\mathbb{S}_n)$
- $\psi(0, r) = \phi(r) \sim c r^n$  as  $r \rightarrow 0^+$  .

In what follows we consider a family of balls  $B(X_j, R_j)$  generated at random, following a strategy described in the next section.

**2.2. Poisson point process.** We consider a Poisson point process  $(X_j, R_j)_j$  in  $\mathbb{S}_n \times \mathbb{R}^+$ , or equivalently  $N(dx, dr)$  a Poisson random measure on  $\mathbb{S}_n \times \mathbb{R}^+$ , with intensity

$$\nu(dx, dr) = f(r) \sigma(dx) dr$$

where  $f$  satisfies the following assumption **A**( $H$ ) for some  $H > 0$ :

- $\text{supp}(f) \subset [0, \pi)$
- $f$  is bounded on any compact subset of  $(0, \pi)$
- $f(r) \sim r^{-n-1+2H}$  as  $r \rightarrow 0^+$  .

Remarks :

- 1) The first condition ensures that no balls of radius  $R_j$  on the sphere self-intersect.
- 2) Since  $\phi(r) \sim c r^n$ ,  $r \rightarrow 0^+$ , the last condition implies that  $\int_{\mathbb{R}^+} \phi(r) f(r) dr < +\infty$ , which means that the mean surface, with respect to  $f$ , of the balls  $B(X_j, R_j)$  is finite.

**2.3. Random field.** Let  $\mathcal{M}$  denote the space of signed measures  $\mu$  on  $\mathbb{S}_n$  with finite total variation  $|\mu|(\mathbb{S}_n)$ , with  $|\mu|$  the total variation measure of  $\mu$ . For any  $\mu \in \mathcal{M}$ , we define

$$(3) \quad X(\mu) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x, r)) N(dx, dr) .$$

Note that the stochastic integral in (3) is well defined since

$$\begin{aligned} \int_{\mathbb{S}_n \times \mathbb{R}^+} |\mu(B(x, r))| f(r) \sigma(dx) dr &\leq \int_{\mathbb{S}_n} \int_{\mathbb{S}_n} \int_{\mathbb{R}^+} \mathbf{1}_{d(x, y) < r} f(r) \sigma(dx) |\mu|(dy) dr \\ &= |\mu|(\mathbb{S}_n) \left( \int_{\mathbb{R}^+} \phi(r) f(r) dr \right) < +\infty . \end{aligned}$$

In the particular case where  $\mu$  is a Dirac measure  $\delta_z$  for some point  $z \in \mathbb{S}_n$  we simply denote

$$(4) \quad X(z) = X(\delta_z) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mathbf{1}_{B(x,r)}(z) N(dx, dr) .$$

The pointwise field  $\{X(z); z \in \mathbb{S}_n\}$  corresponds to the number of random balls  $(X_j, R_j)$  covering each point of  $\mathbb{S}_n$ . Each random variable  $X(z)$  has a Poisson distribution with mean  $\int_{\mathbb{R}^+} \phi(r) f(r) dr$ .

Furthermore for any  $\mu \in \mathcal{M}$ ,

$$\mathbb{E}(X(\mu)) = \mu(\mathbb{S}_n) \left( \int_{\mathbb{R}^+} \phi(r) f(r) dr \right)$$

and

$$\text{Var}(X(\mu)) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x,r))^2 f(r) \sigma(dx) dr \in (0, +\infty] .$$

**2.4. Key lemma.** For  $H > 0$ , we would like to compute the integral

$$\int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x,r))^2 r^{-n-1+2H} \sigma(dx) dr$$

which is a candidate for the variance of an eventually scaling limit. We first introduce  $\mathcal{M}^H$  the set of measures for which the above integral does converge:

$$\mathcal{M}^H = \mathcal{M} \text{ if } 2H < n ; \mathcal{M}^H = \{\mu \in \mathcal{M} ; \mu(\mathbb{S}_n) = 0\} \text{ if } 2H > n .$$

The following lemma deals with the function  $\psi$  prescribed by (2).

**Lemma 2.1.** *Let  $H > 0$  with  $2H \neq n$ . We introduce*

$$\psi^{(H)} = \psi \text{ if } 0 < 2H < n ; \psi^{(H)} = \psi - \sigma(\mathbb{S}_n) \text{ if } 2H > n .$$

*Then for all  $u \in [0, \pi]$ ,*

$$\int_{\mathbb{R}^+} |\psi^{(H)}(u, r)| r^{-n-1+2H} dr < +\infty .$$

*Furthermore, denoting*

$$(5) \quad K_H(u) = \int_{\mathbb{R}^+} \psi^{(H)}(u, r) r^{-n-1+2H} dr$$

*for any  $u$  in  $[0, \pi]$ , we have for all  $\mu \in \mathcal{M}^H$ ,*

$$0 \leq \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x,r))^2 r^{-n-1+2H} \sigma(dx) dr = \int_{\mathbb{S}_n \times \mathbb{S}_n} K_H(d(z, z')) \mu(dz) \mu(dz') < +\infty .$$

**Remark 2.2.**

- 1) *For  $x, y$  in  $\mathbb{S}_n$ , the difference of Dirac measures  $\delta_x - \delta_y$  belongs to  $\mathcal{M}^H$  for any  $H$ .*
- 2) *In the case  $2H > n$ , since any  $\mu \in \mathcal{M}^H$  is centered, the rhs integral is not changed when a constant is added to the kernel  $K_H$ .*
- 3) *This lemma proves that the kernel  $K_H$  defines a covariance function on  $\mathcal{M}^H$ .*

*Proof.* Using the properties of  $\psi$ , we get in the case  $0 < 2H < n$ ,

$$\begin{aligned} 0 \leq \int_{\mathbb{R}^+} \psi(u, r) r^{-n-1+2H} dr &\leq \int_{(0, \pi)} \phi(r) r^{-n-1+2H} dr + \sigma(\mathbb{S}_n) \int_{(\pi, \infty)} r^{-n-1+2H} dr \\ &< +\infty . \end{aligned}$$

In the same vein, in the case  $2H > n$ , we get

$$\begin{aligned} 0 \leq \int_{\mathbb{R}^+} (\sigma(\mathbb{S}_n) - \psi(u, r)) r^{-n-1+2H} dr &\leq \sigma(\mathbb{S}_n) \int_{(0, \pi)} r^{-n-1+2H} dr \\ &< +\infty . \end{aligned}$$

We have just established that there exists a finite constant  $C_H$  such that

$$(6) \quad \forall u \in [0, \pi] , \quad \int_{\mathbb{R}^+} |\psi^{(H)}(u, r)| r^{-n-1+2H} dr \leq C_H .$$

The first statement is proved.

Let us denote for  $\mu \in \mathcal{M}^H$

$$I_H(\mu) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x, r))^2 r^{-n-1+2H} \sigma(dx) dr$$

and start with proving that  $I_H(\mu)$  is finite. We will essentially use Fubini's Theorem in the following lines.

$$\begin{aligned} I_H(\mu) &= \int_{\mathbb{R}^+} \left( \int_{\mathbb{S}_n} \mu(B(x, r))^2 \sigma(dx) \right) r^{-n-1+2H} dr \\ &= \int_{\mathbb{R}^+} \left( \int_{\mathbb{S}_n \times \mathbb{S}_n} \Psi(z, z', r) \mu(dz) \mu(dz') \right) r^{-n-1+2H} dr . \end{aligned}$$

Since  $\mu \in \mathcal{M}^H$  is centered in the case  $2H > n$ , one can change  $\psi$  into  $\psi^{(H)}$  within the previous integral. Then

$$\begin{aligned} I_H(\mu) &\leq \int_{\mathbb{R}^+} \left( \int_{\mathbb{S}_n \times \mathbb{S}_n} |\psi^{(H)}(d(z, z'), r)| |\mu|(dz) |\mu|(dz') \right) r^{-n-1+2H} dr \\ &\leq \int_{\mathbb{S}_n \times \mathbb{S}_n} \left( \int_{\mathbb{R}^+} |\psi^{(H)}(d(z, z'), r)| r^{-n-1+2H} dr \right) |\mu|(dz) |\mu|(dz') \\ &\leq C_H |\mu|(\mathbb{S}_n)^2 < +\infty . \end{aligned}$$

Following the same lines (except for the last one) without the “|” allows the computation of  $I_H(\mu)$  and concludes the proof.  $\square$

An explicit value for the kernel  $K_H$  is available starting from its definition. The point is to compute  $\psi^{(H)}$ . We give in the Appendix a recurrence formula for  $\psi^{(H)}$ , based on the dimension  $n$  of the unit sphere  $\mathbb{S}_n$  (see Lemma 4.1).

**2.5. Scaling.** Let  $\rho > 0$  and  $\lambda$  be any positive function on  $(0, +\infty)$ . We consider the scaled Poisson measure  $N_\rho$  obtained from the original Poisson measure  $N$  by taking the image under the map  $(x, r) \in S \times \mathbb{R}^+ \mapsto (x, \rho r)$  and multiplying the intensity by  $\lambda(\rho)$ . Hence  $N_\rho$  is still a Poisson random measure with intensity

$$\nu_\rho(dx, dr) = \lambda(\rho)\rho^{-1}f(\rho^{-1}r)\sigma(dx)dr.$$

We also introduce the scaled random field  $X_\rho$  defined on  $\mathcal{M}$  by

$$(7) \quad X_\rho(\mu) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x, r)) N_\rho(dx, dr).$$

**Theorem 2.3.** *Let  $H > 0$  with  $2H \neq n$  and let  $f$  satisfy  $\mathbf{A}(H)$ . For all positive functions  $\lambda$  such that  $\lambda(\rho)\rho^{n-2H} \xrightarrow{\rho \rightarrow +\infty} +\infty$ , the limit*

$$\left\{ \frac{X_\rho(\mu) - \mathbb{E}(X_\rho(\mu))}{\sqrt{\lambda(\rho)\rho^{n-2H}}}; \mu \in \mathcal{M}^H \right\} \xrightarrow[\rho \rightarrow +\infty]{fdd} \{W_H(\mu); \mu \in \mathcal{M}^H\}$$

*holds in the sense of finite dimensional distributions of the random functionals. Here  $W_H$  is the centered Gaussian random linear functional on  $\mathcal{M}^H$  with*

$$(8) \quad \text{Cov}(W_H(\mu), W_H(\nu)) = \int_{\mathbb{S}_n \times \mathbb{S}_n} K_H(d(z, z')) \mu(dz) \nu(dz'),$$

*where  $K_H$  is the kernel introduced in Lemma 2.1.*

The theorem can be rephrased in term of the pointwise field  $\{X(z); z \in \mathbb{S}_n\}$  defined in (4).

**Corollary 2.4.** *Let  $H > 0$  with  $2H \neq n$  and let  $f$  satisfy  $\mathbf{A}(H)$ . For all positive functions  $\lambda$  such that  $\lambda(\rho)\rho^{n-2H} \xrightarrow{\rho \rightarrow +\infty} +\infty$ ,*

- *if  $0 < 2H < n$  then*

$$\left\{ \frac{X_\rho(z) - \mathbb{E}(X_\rho(z))}{\sqrt{\lambda(\rho)\rho^{n-2H}}}; z \in \mathbb{S}_n \right\} \xrightarrow[\rho \rightarrow +\infty]{fdd} \{W_H(z); z \in \mathbb{S}_n\}$$

*where  $W_H$  is the centered Gaussian random field on  $\mathbb{S}_n$  with*

$$\text{Cov}(W_H(z), W_H(z')) = K_H(d(z, z')).$$

- *if  $2H > n$  then for any fixed point  $z_0 \in \mathbb{S}_n$ ,*

$$\left\{ \frac{X_\rho(z) - X_\rho(z_0)}{\sqrt{\lambda(\rho)\rho^{n-2H}}}; z \in \mathbb{S}_n \right\} \xrightarrow[\rho \rightarrow +\infty]{fdd} \{W_{H,z_0}(z); z \in \mathbb{S}_n\}$$

*where  $W_{H,z_0}$  is the centered Gaussian random field on  $\mathbb{S}_n$  with*

$$\text{Cov}(W_{H,z_0}(z), W_{H,z_0}(z')) = K_H(d(z, z')) - K_H(d(z, z_0)) - K_H(d(z', z_0)) + K_H(0).$$

*Proof.* of Theorem 2.3.

Let us denote  $n(\rho) := \sqrt{\lambda(\rho)\rho^{n-2H}}$ . The characteristic function of the normalized field  $(X_\rho(\cdot) - \mathbb{E}(X_\rho(\cdot))) / n(\rho)$  is then given by

$$(9) \quad \mathbb{E} \left( \exp \left( i \frac{X_\rho(\mu) - \mathbb{E}(X_\rho(\mu))}{n(\rho)} \right) \right) = \exp \left( \int_{\mathbb{S}_n \times \mathbb{R}^+} G_\rho(x, r) \, d\sigma(dx) \right)$$

where

$$(10) \quad G_\rho(x, r) = \left( e^{i \frac{\mu(B(x, r))}{n(\rho)}} - 1 - i \frac{\mu(B(x, r))}{n(\rho)} \right) \lambda(\rho) \rho^{-1} f(\rho^{-1}r) .$$

We will make use of Lebesgue's Theorem in order to get the limit of  $\int_{\mathbb{S}_n \times \mathbb{R}^+} G_\rho(x, r) \, d\sigma(dx)$  as  $\rho \rightarrow +\infty$ .

On the one hand,  $n(\rho)$  tends to  $+\infty$  so that  $\left( e^{i \frac{\mu(B(x, r))}{n(\rho)}} - 1 - i \frac{\mu(B(x, r))}{n(\rho)} \right)$  behaves like  $-\frac{1}{2} \left( \frac{\mu(B(x, r))}{n(\rho)} \right)^2$ . Together with the assumption **A**( $H$ ), it yields the following asymptotic. For all  $(x, r) \in \mathbb{S}_n \times \mathbb{R}^+$ ,

$$(11) \quad G_\rho(x, r) \xrightarrow{\rho \rightarrow +\infty} -\frac{1}{2} \mu(B(x, r))^2 r^{-n-1+2H} .$$

On the other hand, since  $\frac{|\mu|(B(x, r))}{n(\rho)} \leq |\mu|(\mathbb{S}_n)$  for  $\rho$  large enough, we note that there exists some positive constant  $K$  such that for all  $x, r, \rho$ ,

$$\left| e^{i \frac{\mu(B(x, r))}{n(\rho)}} - 1 - i \frac{\mu(B(x, r))}{n(\rho)} \right| \leq K \left( \frac{\mu(B(x, r))}{n(\rho)} \right)^2 .$$

Therefore

$$|G_\rho(x, r)| \leq K \mu(B(x, r))^2 \rho^{-n-1+2H} f(\rho^{-1}r) .$$

There exists  $C > 0$  such that for all  $r \in \mathbb{R}^+$ ,  $f(r) \leq Cr^{-n-1+2H}$ . Then we get

$$(12) \quad |G_\rho(x, r)| \leq KC \mu(B(x, r))^2 r^{-n-1+2H}$$

where the right hand side is integrable on  $\mathbb{S}_n \times \mathbb{R}^+$  by Lemma 2.1.

Applying Lebesgue's Theorem yields

$$\begin{aligned} \int_{\mathbb{S}_n \times \mathbb{R}^+} G_\rho(x, r) \, d\sigma(dx) \, dr &\xrightarrow{\rho \rightarrow +\infty} -\frac{1}{2} \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x, r))^2 r^{-n-1+2H} \, d\sigma(dx) \, dr \\ &= -\frac{1}{2} \int_{\mathbb{S}_n \times \mathbb{S}_n} K_H(d(z, z')) \, \mu(dz) \mu(dz') . \end{aligned}$$

Hence  $(X_\rho(\mu) - \mathbb{E}(X_\rho(\mu))) / n(\rho)$  converges in distribution to the centered Gaussian random variable  $W(\mu)$  whose variance is equal to

$$\mathbb{E} (W(\mu)^2) = C \int_{\mathbb{S}_n \times \mathbb{S}_n} K_H(d(z, z')) \, \mu(dz) \mu(dz') .$$

By linearity, the covariance of  $W$  satisfies (8). □

**Remark 2.5.**

1) The pointwise limit field  $\{W_H(z); z \in \mathbb{S}_n\}$  in Corollary 2.4 is stationary, i.e. its distribution is invariant under the isometry group of  $\mathbb{S}_n$ , whereas the increments of  $\{W_{H,z_0}(z); z \in \mathbb{S}_n\}$  are distribution invariant under the group of all isometries of  $\mathbb{S}_n$  which keep the point  $z_0$  invariant.

2) When  $0 < H < 1/2$  the Gaussian field  $W_H$  does not coincide with the field introduced in [13] as the spherical fractional Brownian motion on  $\mathbb{S}_n$ .

Indeed, let us have a look at the case  $n = 1$ , it is easy to obtain the following piecewise expression for  $\psi = \psi_1$ :  $\forall (u, r) \in [0, \pi] \times \mathbb{R}^+$ ,

$$\begin{aligned} \psi_1(u, r) &= 0 \quad \text{for } 0 \leq r < u/2 \\ &= 2r - u \quad \text{for } u/2 \leq r \leq \pi - u/2 \\ &= 4r - 2\pi \quad \text{for } \pi - u/2 \leq r \leq \pi \\ &= 2\pi \quad \text{for } r > \pi \end{aligned}$$

and to compute

$$K_H(u) = \frac{1}{H(1-2H)2^{2H}} (2(2H)^{2H} - u^{2H} - (2\pi - u)^{2H}) ,$$

Actually, we compute the variance of the increments of  $W_H$

$$\begin{aligned} \mathbb{E}(W_H(z) - W_H(z'))^2 &= 2K_H(0) - 2K_H(d(z, z')) \\ &= \frac{2}{H(1-2H)2^{2H}} [d^{2H}(z, z') + (2\pi - d(z, z'))^{2H} - (2\pi)^{2H}]. \end{aligned}$$

The spherical fractional Brownian motion  $B_H$ , introduced in [13], satisfies

$$\mathbb{E}(B_H(z) - B_H(z'))^2 = d^{2H}(z, z').$$

Even up to a constant, processes  $W_H$  and  $B_H$  are clearly different. The Euclidean situation is therefore different. Indeed, [3], the variance of the increments of the corresponding field  $W_H$  is proportional to  $|z - z'|^{2H}$ .

### 3. LOCAL SELF-SIMILAR BEHAVIOR

We wonder whether the limit field  $W_H$  obtained in the previous section satisfies a local asymptotic self-similar (*lass*) property. More precisely we will let a “dilation” of order  $\varepsilon$  act on  $W_H$  near a fixed point  $A$  in  $\mathbb{S}_n$  and as in [15], up to a renormalization factor, we look for an asymptotic behavior as  $\varepsilon$  goes to 0. An  $H$ -self-similar tangent field  $T_H$  is expected. Recall that  $W_H$  is defined on a subspace  $\mathcal{M}^H$  of measures on  $\mathbb{S}_n$ , so that  $T_H$  will be defined on a subspace of measures on the tangent space  $\mathcal{T}_A\mathbb{S}_n$  of  $\mathbb{S}_n$ .

**3.1. Dilation.** Let us fix a point  $A$  in  $\mathbb{S}_n$  and consider  $\mathcal{T}_A\mathbb{S}_n$  the tangent space of  $\mathbb{S}_n$  at  $A$ . It can be identified with  $\mathbb{R}^n$  and  $A$  with the null vector of  $\mathbb{R}^n$ .

Let  $1 < \delta < \pi$ . The exponential map at point  $A$ , denoted by  $\exp$ , is a diffeomorphism between the Euclidean ball  $\{y \in \mathbb{R}^n, \|y\| < \delta\}$  and  $\overset{\circ}{B}(A, \delta) \subset \mathbb{S}_n$ , where  $\|\cdot\|$  denotes the



Euclidean norm in  $\mathbb{R}^n$  and  $\overset{\circ}{B}(A, \delta)$  the open ball with center  $A$  and radius  $\delta$  in  $\mathbb{S}_n$ . Furthermore for all  $y, y' \in \mathbb{R}^n$  such that  $\|y\|, \|y'\| < \delta$ ,

$$d(A, \exp y) = \|y\| \quad \text{and} \quad d(\exp y, \exp y') \leq \|y - y'\| .$$

We refer to [10] for precisions on the exponential map.

Let  $\tau$  be a signed measure on  $\mathbb{R}^n$ . We define the dilated measure  $\tau_\varepsilon$  by

$$\forall B \in \mathcal{B}(\mathbb{R}^n) \quad \tau_\varepsilon(B) = \tau(B/\varepsilon)$$

and then map it by the application  $\exp$ , defining the measure  $\mu_\varepsilon = \exp^* \tau_\varepsilon$  on  $\overset{\circ}{B}(A, \delta)$  by

$$(13) \quad \forall C \in \mathcal{B}(\overset{\circ}{B}(A, \delta)) \quad \mu_\varepsilon(C) = \exp^* \tau_\varepsilon(C) = \tau_\varepsilon(\exp^{-1}(C)).$$

We then consider the measure  $\mu_\varepsilon$  as a measure on the whole sphere  $\mathbb{S}_n$  with support included in  $\overset{\circ}{B}(A, \delta)$ .

At last, we define the dilation of  $W_H$  within a “neighborhood of  $A$ ” by the following procedure. For any finite measure  $\tau$  on  $\mathbb{R}^n$ , we consider  $\mu_\varepsilon = \exp^* \tau_\varepsilon$  as defined by (13) and compute  $W_H(\mu_\varepsilon)$ . We will establish the convergence in distribution of  $\varepsilon^{-H} W_H(\exp^* \tau_\varepsilon)$  for any  $\tau$  in an appropriate space of measures on  $\mathbb{R}^n$ . Since  $W_H(\mu_\varepsilon)$  is Gaussian, we will focus on its variance.

**3.2. Asymptotic of the kernel  $K_H$ .** For  $0 < H < 1/2$ , we already mentioned that the kernel  $K_H(0) - K_H(u)$  is not proportional to  $u^{2H}$ . As a consequence, one cannot expect  $W_H$  to be self-similar. Nevertheless, as we are looking for an asymptotic local self-similarity, only the behavior of  $K_H$  near zero is relevant. Actually we will establish that, up to a constant,  $K_H(0) - K_H(u)$  behaves like  $u^{2H}$  when  $u \rightarrow 0^+$ .

**Lemma 3.1.** *Let  $0 < H < 1/2$ . The kernel  $K_H$  defined by (5) satisfies*

$$K_H(u) = K_1 - K_2 u^{2H} + o(u^{2H}), \quad u \rightarrow 0^+$$

where  $K_1 = K_H(0)$  and  $K_2$  are nonnegative constants.

*Proof.* Let us state that the assumption  $H < 1/2$  implies  $H < n/2$  so that in that case  $K_H$  is prescribed by

$$K_H(u) = \int_{\mathbb{R}^+} \psi(u, r) r^{-n-1+2H} dr, \quad u \in [0, \pi] .$$

We note that  $K_H(0) < +\infty$  since  $\psi(0, r) \sim cr^n$  as  $r \rightarrow 0^+$  and  $\psi(0, r) = \sigma(\mathbb{S}_n)$  for  $r > \pi$ . Then, subtracting  $K_H(0)$  and remarking that  $\psi(0, r) = \psi(u, r) = \sigma(\mathbb{S}_n)$  for  $r > \pi$ , we

write

$$\begin{aligned} K_H(0) - K_H(u) &= \int_0^\pi (\psi(0, r) - \psi(u, r)) r^{-n-1+2H} dr \\ &= \int_0^\delta (\psi(0, r) - \psi(u, r)) r^{-n-1+2H} dr \\ &\quad + \int_\delta^\pi (\psi(0, r) - \psi(u, r)) r^{-n-1+2H} dr \end{aligned}$$

where we recall that  $\delta \in (1, \pi)$  is such that the exponential map is a diffeomorphism between  $\{\|y\| < \delta\} \subset \mathbb{R}^n$  and  $\mathring{B}(A, \delta) \subset \mathbb{S}_n$ .

The second term is of order  $u$ , and therefore is negligible with respect to  $u^{2H}$ , since  $\psi$  is clearly Lipschitz on the compact interval  $[\delta, \pi]$ .

We now focus on the first term. Performing the change of variable  $r \mapsto r/u$ , we write it as

$$\int_0^\delta (\psi(0, r) - \psi(u, r)) r^{-n-1+2H} dr = u^{2H} \int_{\mathbb{R}^+} \Delta(u, r) r^{-n-1+2H} dr ,$$

where

$$\Delta(u, r) := \mathbf{1}_{ur < \delta} u^{-n} (\psi(u, ur) - \psi(0, ur)) .$$

Their only remains to prove that  $\int_{\mathbb{R}^+} \Delta(u, r) r^{-n-1+2H} dr$  admits a finite limit  $K_2$  as  $u \rightarrow 0^+$ . We will use Lebesgue's Theorem and start with establishing the simple convergence of  $\Delta(u, r)$  for any given  $r \in \mathbb{R}^+$ .

We fix a unit vector  $\mathbf{v}$  in  $\mathbb{R}^n$  and a point  $A' = \exp \mathbf{v}$  in  $\mathbb{S}_n$ . We then consider for any  $u \in (0, \delta)$ , the point  $A'_u := \exp(u\mathbf{v}) \in \mathbb{S}_n$  located on the geodesic between  $A$  and  $A'$  such that  $d(A, A'_u) = \|u\mathbf{v}\| = u$ . We can then use (1) and (2) to write

$$\psi(u, \cdot) = \Psi(A, A'_u, \cdot) = \int_{\mathbb{S}_n} \mathbf{1}_{d(A, z) < \cdot} \mathbf{1}_{d(A'_u, z) < \cdot} d\sigma(z)$$

and

$$\psi(0, \cdot) = \Psi(A, A, \cdot) = \int_{\mathbb{S}_n} \mathbf{1}_{d(A, z) < \cdot} d\sigma(z)$$

in order to express  $\Delta(u, r)$  as

$$\Delta(u, r) = \mathbf{1}_{ur < \delta} u^{-n} \int_{\mathbb{S}_n} \mathbf{1}_{d(A, z) < ur} \mathbf{1}_{d(A'_u, z) > ur} d\sigma(z) .$$

Since  $ur < \delta$  the above integral runs on  $\mathring{B}(A, ur) \subset \mathring{B}(A, \delta)$  and we can perform the exponential change of variable to get

$$\begin{aligned} \Delta(u, r) &= \mathbf{1}_{ur < \delta} u^{-n} \int_{\mathbb{R}^n} \mathbf{1}_{\|y\| < ur} \mathbf{1}_{d(\exp(u\mathbf{v}), \exp(y)) > ur} d\sigma(\exp(y)) \\ &= \mathbf{1}_{ur < \delta} \int_{\mathbb{R}^n} \mathbf{1}_{\|y\| < r} \mathbf{1}_{d(\exp(u\mathbf{v}), \exp(uy)) > ur} \tilde{\sigma}(uy) dy . \end{aligned}$$

In the last integral, the image by  $\exp$  of the surface measure  $d\sigma(\exp(y))$  is written as  $\tilde{\sigma}(y)dy$  where  $dy$  denotes the Lebesgue measure on  $\mathbb{R}^n$ .

We use the fact that  $d(\exp(ux), \exp(ux')) \sim u\|x - x'\|$  as  $u \rightarrow 0^+$  to get the following limit for the integrand

$$\mathbf{1}_{d(\exp(u\mathbf{v}), \exp(uy)) < ur} \tilde{\sigma}(uy) \longrightarrow \mathbf{1}_{\|\mathbf{v}-y\| > r} \tilde{\sigma}(0) .$$

Since the integrand is clearly dominated by

$$\|\sigma\|_\infty := \sup\{\tilde{\sigma}(y), \|y\| \leq \delta\} ,$$

Lebesgue's Theorem yields for all  $r \in \mathbb{R}^+$ ,

$$\Delta(u, r) \longrightarrow \tilde{\sigma}(0) \int_{\mathbb{R}^n} \mathbf{1}_{\|y\| < r} \mathbf{1}_{\|\mathbf{v}-y\| > r} dy .$$

We recall that  $d(\exp x, \exp x') \leq \|x - x'\|$  for all  $x, x' \in \mathbb{R}^n$  with norm less than  $\delta$ . Therefore for all  $u$ ,

$$\Delta(u, r) \leq \|\sigma\|_\infty \int_{\mathbb{R}^n} \mathbf{1}_{\|y\| < r} \mathbf{1}_{\|\mathbf{v}-y\| > r} dy$$

where the right hand side belongs to  $L^1(\mathbb{R}^+, r^{-n-1+2H} dr)$  (see [2] Lemma A.2).

Using Lebesgue's Theorem for the last time, we obtain

$$\int_{\mathbb{R}^+} \Delta(u, r) r^{-n-1+2H} dr \xrightarrow{u \rightarrow 0^+} K_2$$

where

$$K_2 = \tilde{\sigma}(0) \int_{\mathbb{R}^n \times \mathbb{R}^+} \mathbf{1}_{\|y\| < r} \mathbf{1}_{\|\mathbf{v}-y\| > r} r^{-n-1+2H} dy dr \in (0, +\infty) .$$

□

Let us remark that the proof makes it clear that the case  $H > 1/2$  is dramatically different. The kernel  $K_H(0) - K - H(u)$  behaves like  $u$  near zero and loses its  $2H$  power.

### 3.3. Main result.

Let  $0 < H < 1/2$ . We consider the following space of measures on  $\mathcal{T}_A \mathbb{S}_n \cong \mathbb{R}^n$

$$\begin{aligned} \mathfrak{M}^H &= \{\text{measures } \tau \text{ on } \mathbb{R}^n \text{ with finite total variation such that} \\ &\quad \tau(\mathbb{R}^n) = 0 \text{ and } \int_{\mathbb{R}^n \times \mathbb{R}^n} \|x - x'\|^{2H} |\tau|(dx) |\tau|(dx') < +\infty\} . \end{aligned}$$

For any measure  $\tau \in \mathfrak{M}^H$ , we compute the variance of  $W_H(\mu_\varepsilon)$  where  $\mu_\varepsilon = \exp^* \tau_\varepsilon$  is defined by (13).

By Lemma 2.1, since  $\mu_\varepsilon$  belongs to  $\mathcal{M} = \mathcal{M}^H$  in the case  $H < 1/2$ ,

$$\text{var}(W_H(\mu_\varepsilon)) = \int_{B(A, \delta) \times B(A, \delta)} K_H(d(z, z')) \mu_\varepsilon(dz) \mu_\varepsilon(dz') .$$

Performing an exponential change of variable followed by a dilation in  $\mathbb{R}^n$ , we get

$$\begin{aligned} \text{var}(W_H(\mu_\varepsilon)) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathbf{1}_{\|y\| < \delta} \mathbf{1}_{\|y'\| < \delta} K_H(d(\exp(y), \exp(y'))) \tau_\varepsilon(dy) \tau_\varepsilon(dy') \\ &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathbf{1}_{\|x\| < \delta/\varepsilon} \mathbf{1}_{\|x'\| < \delta/\varepsilon} K_H(d(\exp(\varepsilon x), \exp(\varepsilon x'))) \tau(dx) \tau(dx'). \end{aligned}$$

Denoting  $\widetilde{K}_H(u) = K_H(u) - K_H(0)$ ,

$$\begin{aligned} \text{var}(W_H(\mu_\varepsilon)) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathbf{1}_{\|x\| < \delta/\varepsilon} \mathbf{1}_{\|x'\| < \delta/\varepsilon} \widetilde{K}_H(d(\exp(\varepsilon x), \exp(\varepsilon x'))) \tau(dx) \tau(dx') \\ &\quad + K_H(0) \tau(\{\|x\| < \delta/\varepsilon\})^2. \end{aligned}$$

Let us admit for a while that

$$(14) \quad \frac{\tau(\{\|x\| < \delta/\varepsilon\})^2}{\varepsilon^{2H}} \xrightarrow{\varepsilon \rightarrow 0^+} 0.$$

Then, applying Lebesgue's Theorem with the convergence argument on  $\widetilde{K}_H$  obtained in Lemma 3.1, yields

$$(15) \quad \frac{\text{var}(W_H(\mu_\varepsilon))}{\varepsilon^{2H}} \xrightarrow{\varepsilon \rightarrow 0^+} -K_2 \int_{\mathbb{R}^n \times \mathbb{R}^n} \|x - x'\|^{2H} \tau(dx) \tau(dx').$$

Let us now establish (14) where we recall that  $\tau$  is any measure in  $\mathfrak{M}^H$ . In particular, the total mass of  $\tau$  is zero so that

$$\begin{aligned} \frac{\tau(\{\|x\| < \delta/\varepsilon\})}{\varepsilon^H} &= - \frac{\tau(\{\|x\| > \delta/\varepsilon\})}{\varepsilon^H} \\ &= - \int_{\mathbb{R}^n} \varepsilon^{-H} \mathbf{1}_{\|x\| > \delta/\varepsilon} \tau(dx). \end{aligned}$$

For any fixed  $x \in \mathbb{R}^n$ ,  $\varepsilon^{-H} \mathbf{1}_{\|x\| > \delta/\varepsilon}$  is zero when  $\varepsilon$  is small enough. Moreover  $\varepsilon^{-H} \mathbf{1}_{\|x\| > \delta/\varepsilon}$  is dominated by  $\delta^{-H} \|x\|^H$  which belongs to  $L^1(\mathbb{R}^n, |\tau|(dx))$  since  $\tau$  belongs to  $\mathfrak{M}^H$ . Lebesgue's Theorem applies once more.

We deduce from asymptotic (15) the following theorem.

**Theorem 3.2.** *Let  $0 < H < 1/2$ . The limit*

$$\frac{W_H(\exp^* \tau_\varepsilon)}{\varepsilon^H} \xrightarrow[\varepsilon \rightarrow 0^+]{fdd} T_H(\tau)$$

*holds for all  $\tau \in \mathfrak{M}^H$ , in the sense of finite dimensional distributions of the random functionals. Here  $T_H$  is the centered Gaussian random linear functional on  $\mathfrak{M}^H$  with*

$$(16) \quad \text{Cov}(T_H(\tau), T_H(\tau')) = -K_2 \int_{\mathbb{R}^n \times \mathbb{R}^n} \|x - x'\|^{2H} \tau(dx) \tau'(dx'),$$

As for Theorem 2.3, Theorem 3.2 can be rephrased in terms of pointwise fields. Indeed,  $\delta_x - \delta_O$  belongs to  $\mathfrak{M}^H$  for all  $x$  in  $\mathbb{R}^n$ . Let us apply Theorem 3.2 with  $\tau = \delta_x - \delta_O$ . Then  $T_H(\delta_x - \delta_O)$  has the covariance

$$\text{Cov}(T_H(\delta_x - \delta_O), T_H(\delta_{x'} - \delta_O)) = K_2(\|x\|^{2H} + \|x'\|^{2H} - \|x - x'\|^{2H}),$$

and the field  $\{T_H(\delta_x - \delta_O); x \in \mathbb{R}^n\}$  is a Euclidean fractional Brownian field.

#### 4. COMPARATIVE ANALYSIS

In this section, we aim to discuss the differences and the analogies between the Euclidean and the spherical case.

Let us first be concerned with the existence of a scaling limit random field. The variance of this limit field should be

$$\mathbf{V} = \int_{\mathbb{M}_n} \int_{\mathbb{R}^+} \mu(B(x, r))^2 \sigma(dx) r^{-n-1+2H} dr,$$

where  $\mathbb{M}_n$  is the  $n$ -dimensional corresponding surface with its surface measure  $\sigma$ . When speaking of the Euclidean case  $\mathbb{M}_n = \mathbb{R}^n$  we refer to [3]. In the present paper, we studied the case  $\mathbb{M}_n = \mathbb{S}_n$ . Moreover, in this discussion, the hyperbolic case  $\mathbb{M}_n = \mathbb{H}_n = \{(x_i)_{1 \leq i \leq n+1} \in \mathbb{R}^{n+1}; x_{n+1}^2 - \sum_{1 \leq i \leq n} x_i^2 = 1, x_{n+1} \geq 1\}$  is evoked.

In the Euclidean case, the random fields are defined on the space of measures with vanishing total mass. So let us first consider measures  $\mu$  such that  $\mu(\mathbb{M}_n) = 0$ . Hence, whatever the surface  $\mathbb{M}_n$ , the integral  $\mathbf{V}$  involves the integral of the surface of the symmetric difference between two balls of same radius  $r$ . As  $r$  goes to infinity, three different behaviors emerge.

- $\mathbb{M}_n = \mathbb{S}_n$ : this surface vanishes
- $\mathbb{M}_n = \mathbb{R}^n$ : the order of magnitude of this surface is  $r^{n-1}$
- $\mathbb{M}_n = \mathbb{H}_n$ : the surface grows exponentially

The consequences are the following.

- $\mathbb{M}_n = \mathbb{S}_n$ : any positive  $H$  is admissible
- $\mathbb{M}_n = \mathbb{R}^n$ : the range of admissible  $H$  is  $(0, 1/2)$
- $\mathbb{M}_n = \mathbb{H}_n$ : no  $H$  is admissible

In the Euclidean case, the restriction  $\mu(\mathbb{R}^n) = 0$  is mandatory whereas it is unnecessary in the spherical case for  $H < n/2$ . Indeed the integral  $\mathbf{V}$  is clearly convergent.

Let us now discuss the (local) self-similarity of the limit field. Of course, we no longer consider the hyperbolic case.

- $\mathbb{M}_n = \mathbb{R}^n$ : dilating a ball is a homogeneous operation. Therefore, the limit field is self-similar.
- $\mathbb{M}_n = \mathbb{S}_n$ : dilation is no longer homogeneous. Only local self-similarity can be expected. The natural framework of this local self-similarity is the tangent bundle, where the situation is Euclidean. Hence we have to come back to the restricting condition  $H < 1/2$ .

#### APPENDIX

##### Recurrence formula for the $\psi_n$ 's.

Recall that the functions  $\psi_n$ 's are defined by (1) and (2)

$$\psi_n(u, r) = \Psi_n(M, M', r) = \int_{\mathbb{S}_n} \mathbf{1}_{d(M, N) < r} \mathbf{1}_{d(M', N) < r} d\sigma_n(N), \quad (u, r) \in [0, \pi] \times \mathbb{R}^+,$$

for any pair  $(M, M')$  in  $\mathbb{S}_n$  such that  $d(M, M') = u$ . Here  $\sigma_n$  stands for the surface measure on  $\mathbb{S}_n$ .

**Lemma 4.1.** *The family of functions  $\psi_n, n \geq 2$  satisfies the following recursion:*  
 $\forall (u, r) \in [0, \pi] \times \mathbb{R}^+,$

$$\psi_n(u, r) = \int_{-\sin r}^{\sin r} (1 - a^2)^{n/2} \psi_{n-1} \left( u, \arccos \left( \frac{\cos r}{\sqrt{1 - a^2}} \right) \right) da.$$

*Proof.* An arbitrary point of  $\mathbb{S}_n$  is parameterized either in Cartesian coordinates,  $(x_i)_{1 \leq i \leq n+1}$ , or in spherical ones

$$(\phi_i)_{1 \leq i \leq n} \in [0, \pi)^{n-1} \times [0, 2\pi)$$

with

$$\begin{aligned} x_1 &= \cos \phi_1 \\ x_2 &= \sin \phi_1 \cos \phi_2 \\ x_3 &= \sin \phi_1 \sin \phi_2 \cos \phi_3 \\ &\dots \\ x_n &= \sin \phi_1 \sin \phi_2 \dots \sin \phi_{n-1} \cos \phi_n \\ x_{n+1} &= \sin \phi_1 \sin \phi_2 \dots \sin \phi_{n-1} \sin \phi_n \end{aligned}$$

Let  $M$  be the point  $(\phi_i)_{1 \leq i \leq n} = (0, \dots, 0)$ . One can write the ball  $B_n(M, r)$  of radius  $r$ , which is a spherical cap on  $\mathbb{S}_n$  with opening angle  $r$  as follows,

$$B_n(M, r) = \{(\phi_i)_{1 \leq i \leq n} \in \mathbb{S}_n ; \phi_1 \leq r\}$$

or in Cartesian coordinates

$$B_n(M, r) = \{(x_i)_{1 \leq i \leq n+1} \in \mathbb{S}_n ; x_1 \geq \cos r\}.$$

Let  $a \in (-1, 1)$  and let  $P_a$  be the hyperplane of  $\mathbb{R}^{n+1}$  defined by  $x_{n+1} = a$ . Let us consider the intersection  $P_a \cap B_n(M, r)$ .

- If  $1 - a^2 < \cos^2 r$  then  $P_a \cap B_n(M, r) = \emptyset$ .
- If  $1 - a^2 \geq \cos^2 r$  then

$$\begin{aligned} P_a \cap B_n(M, r) &= \{(x_i)_{1 \leq i \leq n+1} \in \mathbb{S}_n ; x_1 \geq \cos r \text{ and } x_{n+1} = a\} \\ &= \{(x_i)_{1 \leq i \leq n} \in \mathbb{R}^n ; x_1 \geq \cos r \text{ and } \sum_{1 \leq i \leq n} x_i^2 = 1 - a^2\} \times \{a\}. \end{aligned}$$

In other words, denoting  $\mathbb{S}_{n-1}(R)$  the  $(n-1)$ -dimensional sphere of radius  $R$ ,

$$P_a \cap B_n(M, r) = B_{n-1, \sqrt{1-a^2}}(M(a), r(a)) \times \{a\}$$

where  $B_{n-1, \sqrt{1-a^2}}(M(a), r(a))$  is the spherical cap on  $\mathbb{S}_{n-1}(\sqrt{1-a^2})$ , centered at  $M(a) = (\sqrt{1-a^2}, 0, \dots, 0)$  and with opening angle  $r(a) = \arccos \left( \frac{\cos r}{\sqrt{1-a^2}} \right)$ .

Let now  $M'$  be defined in spherical coordinates by  $(\phi_i)_{1 \leq i \leq n} = (u, 0, \dots, 0)$ , so that  $d(M, M') = u$ . The intersection  $P_a \cap B_n(M', r)$  is the map of  $P_a \cap B_n(M, r)$  by the rotation of angle  $u$  and center  $C$  in the plane  $x_3 = \dots = x_{n+1} = 0$ . So

- if  $1 - a^2 < \cos^2 r$  then  $P_a \cap B_n(M', r) = \emptyset$ .
- if  $1 - x_0^2 \geq \cos^2 r$  then

$$P_a \cap B_n(M', r) = B_{n-1, \sqrt{1-a^2}}(M'(a), r(a)) \times \{a\}$$

where the  $(n-1)$ -dimensional spherical cap  $B_{n-1, \sqrt{1-a^2}}(M'(a), r(a))$  is now centered at  $M'(a) = (\sqrt{1-a^2} \cos u, \sqrt{1-a^2} \sin u, 0, \dots, 0)$ .

We define  $\psi_{n-1, R}(u, r)$  as the intersection surface of two spherical caps on  $\mathbb{S}_{n-1}(R)$ , whose centers are at a distance  $Ru$  and with the same opening angle  $r$ .

By homogeneity, this leads to

$$\psi_{n-1, R}(u, r) = R^{n-1} \psi_{n-1, 1}(u, r) = R^n \psi_{n-1}(u, r).$$

The surface measure  $\sigma_n$  of  $\mathbb{S}_n$  can be written as

$$d\sigma_n(x_1, \dots, x_n, a) = \sqrt{1-a^2} d\sigma_{n-1, \sqrt{1-a^2}}(x_1, \dots, x_n) \times da$$

where  $\sigma_{n-1, R}$  is the surface measure of  $\mathbb{S}_{n-1}(R)$ .

We then obtain

$$\begin{aligned} \psi_n(u, r) &= \int_{-1}^1 \mathbf{1}_{1-a^2 \geq \cos^2 r} \psi_{n-1, \sqrt{1-a^2}}(u, \arccos\left(\frac{\cos r}{\sqrt{1-a^2}}\right)) \sqrt{1-a^2} da \\ &= \int_{-\sin r}^{\sin r} (\sqrt{1-a^2})^n \psi_{n-1}(u, \arccos\left(\frac{\cos r}{\sqrt{1-a^2}}\right)) da, \end{aligned}$$

and Lemma 4.1 is proved.  $\square$

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ANNE ESTRADE, MAP5 UMR CNRS 8145, UNIVERSITÉ PARIS DESCARTES, 45 RUE DES SAINTS-PÈRES, F-75270 PARIS CEDEX 06 & ANR-05-BLAN-0017

*E-mail address:* `anne.estrade@parisdescartes.fr`

JACQUES ISTAS, LABORATOIRE JEAN KUNTZMANN, UNIVERSITÉ DE GRENOBLE ET CNRS, F-38041 GRENOBLE CEDEX 9

*E-mail address:* `Jacques.Istas@imag.fr`